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BMN Operators for $\mathcal{N} = 1$ Superconformal Yang-Mills Theories and Associated String Backgrounds

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We study a class of near-BPS operators for a complex 2-parameter family of $\mathcal{N} = 1$ superconformal Yang-Mills theories that can be obtained by a Leigh-Strassler deformation of $\mathcal{N} = 4$ SYM theory. We identify these operators in the large N and large R-charge limit and compute their exact scaling dimensions using $\mathcal{N} = 1$ superspace methods. From these scaling dimensions we attempt to reverse-engineer the light-cone worldsheet theory that describes string propagation on the Penrose limit of the dual geometry.

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1. Introduction

According to the AdS/CFT correspondence [1,2,3] string theory on spaces of the form $AdS_d \times \mathcal{M}^{10-d}$ is dual to a conformal field theory that lives on the $(d-1)$-dimensional boundary of AdS_d. Several examples of this correspondence have been studied so far. From the CFT point of view new conformal or nonconformal examples can be obtained by deforming the gauge theory action with a local operator $\mathcal{O}$

$$S \rightarrow S + h \int d^d x \mathcal{O}(x). \tag{1.1}$$

Usually such deformations break some or all of the initial supersymmetry and in most cases it is a nontrivial task to determine how this deformation reflects itself on the string theory side. When $\mathcal{O}$ is a relevant operator the deformation breaks conformal invariance and the RG flow can lead to an interacting IR fixed point. On the gravity side such a deformation yields a complicated space with a running dilaton that interpolates between two AdS geometries of the form $AdS \times \mathcal{M}_{UV}$ and $AdS \times \mathcal{M}_{IR}$. When $\mathcal{O}$ is exactly marginal, conformal invariance remains as a true symmetry of the theory and the dual geometry takes the form $AdS \times \mathcal{M}_h$, with $\mathcal{M}_h$ a compact deformed version of the original manifold $\mathcal{M}$.

We are interested in type IIB string theory on $AdS_5 \times S^5$ and deformations of its dual $\mathcal{N} = 4$ SYM. Many interesting papers have been written on this subject. For example, non-supersymmetric deformations were discussed in [4]. Exactly marginal and relevant deformations that preserve $\mathcal{N} = 1$ supersymmetry were discussed in [5-12] (for a brief review see [13], section 4.3). A relevant perturbation that leads to a confining gauge theory was discussed in [14]. All these cases were considered in the large t'Hooft limit, where supergravity is reliable.

Here we want to discuss a certain class of $\mathcal{N} = 1$ superconformal Yang-Mills theories that can be obtained by a Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory [15,12]. Our analysis focuses on the properties of various near-BPS gauge theory operators with large R-charge. These operators were considered recently by the authors of [16], who also proposed an exact correspondence between such gauge theory operators and string states on the Penrose limit of the $AdS_5 \times S^5$ geometry. Working solely within the deformed gauge theory we use $\mathcal{N} = 1$ superspace methods, in a fashion first proposed in [17], to determine their exact scaling dimensions for any value of the perturbing parameters and at strong t' Hooft coupling. In general, these operators are not protected, since they do not fall into short multiplets of the $SU(2,2|1)$ superconformal group and they obtain anomalous dimensions as one moves away from the weakly coupled $\mathcal{N}=4$ SYM point. Scaling dimensions of such non-protected operators are expected (already at the $\mathcal{N}=4$ point) to diverge at strong t' Hooft coupling as $(g_{YM}^2 N)^{1/4}$, but as a special property of the large R-charge limit of [16] they approach a finite value at strong t' Hooft coupling.

With these operators at hand and following the spirit of the proposal in [16], we can further ask for a light-cone worldsheet theory, whose spectrum reproduces the scaling dimensions we found. Once the worldsheet theory has been determined, we can further attempt to read off the dual string theory background. We find that such a process does not result in a unique background in the infinite R-charge limit. There is, however, a unique one which exhibits supersymmetry enhancement from sixteen to twenty-four supersymmetries.

This reverse-engineering of a string theory from data available in gauge theory would provide, in general, a very powerful method for uncovering further examples of gauge-gravity duals and one would like to have, if possible, a generic prescription to achieve it. In this paper we use the very special properties of the correspondence proposed in [16]; in order to achieve a similar task in a more generic situation one would first have to understand better how to extend this correspondence to finite R-charge and in cases without conformal invariance and/or no supersymmetry.

For the $\mathcal{N}=4$ SYM theory at large R-charge, we should focus on the Penrose limit of $AdS_5 \times S^5$ [18,19,20,21]. This limit leads to a maximally supersymmetric background with metric

$$ds^2 = -4dx^+dx^- + \sum_{i=1}^{8}(dr^i dr^i - r^i r^i (dx^+)^2), \tag{1.2}$$

and constant R-R 5-form flux

$$F_{+1234} = F_{+5678} = \text{const.} \tag{1.3}$$

One of the merits of this background is the exact solvability of the associated worldsheet theory in the light-cone Green-Schwarz formalism, where it simply reduces to a sum of massive oscillators [22,23]. On the gauge theory side the Hilbert space of the $\mathcal{N}=4$ SYM is suitably truncated to states with large scaling dimension $\Delta \sim \sqrt{N}$ and large $U(1)_R$ R-charge $J \sim \sqrt{N}$, while the difference $(\Delta - J)$ is kept fixed and small. A correspondence between such states and on-shell states of string theory in the bulk pp-wave background was proposed by Berenstein, Maldacena and Nastase (BMN) in [16] and as a check the scaling dimensions on both sides were computed and were found to agree. Further checks of this correspondence (and beyond the planar limit) were performed in [17,24-30].

We can obtain a whole moduli space of $\mathcal{N}=1$ SYM theories by perturbing the $\mathcal{N}=4$ Lagrangian by a superpotential that breaks the $SU(4)_R$ R-symmetry group to a diagonal $U(1)_R$ under which all six of the Higgs fields are charged. This $U(1)_R$ is different from the one that was considered in [16] and for that reason it is useful to present a slight variant of that discussion for the $\mathcal{N}=4$ theory. We perform the Penrose limit of $AdS_5 \times S^5$ around the appropriate geodesic and repeat the BMN analysis to rephrase the correspondence between string theory and gauge theory. We find that the resulting pp-wave limit has a metric of the form

$$ds^2 = -4dx^+dx^- + 4\mu y_1 dx_1 dx^+ + 4\mu y_2 dx_2 dx^+ - \mu^2 \vec{r}^2 (dx^+)^2 + d\vec{r}^2 + d\vec{y}^2 + d\vec{x}^2, \quad (1.4)$$

and a 5-form field strength of the form

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5, \quad \mathcal{F}_5 \sim \mu dx^+ \wedge dy^1 \wedge dx^1 \wedge dy^2 \wedge dx^2. \quad (1.5)$$

μ is a mass parameter that can be scaled out through the rescaling $x^+ \rightarrow x^+/\mu$ and $x^- \rightarrow \mu x^-$. In the rest of the paper it is set to one. The Green-Schwarz light-cone worldsheet action includes four massive harmonic oscillators as in [16] and a Landau part that corresponds to the action of a charged particle moving in the presence of a constant magnetic field. This action is again exactly solvable and the string spectrum is known. In fact, after a suitable x^+-dependent change of coordinates the magnetic background of (1.4) transforms into (1.2) [31]. On the gauge theory side, the Penrose limit restricts the $\mathcal{N}=4$ SYM Hilbert space into the same subsector as the one that appears in [16], but the R-charge assignments are now different. As a result, the BMN correspondence involves at each level an infinite degeneracy. On the string theory side this is the usual infinite degeneracy of Landau levels.

The organization of this paper is the following. In section 2, we discuss in detail the Penrose limit of interest and derive the resulting geometry at the $\mathcal{N}=4$ point. We consider string propagation on this geometry and review the associated string spectra. Then, we focus on the gauge theory side and construct the string oscillators from the appropriate gauge invariant SYM operators in the spirit of [16]. This analysis is useful, because it clarifies some characteristics of the BMN correspondence under a different R-charge assignment and it hints as to what may be expected to change or remain the same as we deform away from the $\mathcal{N}=4$ point. In section 3 we briefly review the 2-complex parameter class of exactly marginal deformations of the $\mathcal{N}=4$ SYM theory that will be the main focus of our analysis. This class of theories was introduced in [15] and further studied in connection with AdS/CFT in [6,9,10,11,12] . We proceed to determine the properties of the BMN operators after the Leigh-Strassler deformations using $\mathcal{N}=1$ superspace techniques. We write down appropriate two-point functions of these operators and deduce their exact scaling dimensions in a fashion similar to [17]. As a further check of this result, we perform a perturbative calculation to verify in leading order that the scaling dimensions depend on the deforming parameters as expected. In section 4 we use the available gauge theory data to reconstruct the worldsheet action for string propagation in the Penrose limit of the dual geometry and provide a detailed analysis of the supersymmetries preserved by the associated pp-wave. In section 5 we present our conclusions and suggest directions for further research.

2. A "magnetic" pp-wave limit of $AdS_5 \times S^5$ and its gauge theory dual

2.1. The Penrose limit

Let us start with the $AdS_5 \times S^5$ metric

$$ds^2 = R^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2) \tag{2.1}$$

and write explicitly the solid angle $d\Omega_3'^2$ in S^5 as

$$d\Omega_3'^2 = \cos^2\phi_1 d\phi_2^2 + d\phi_1^2 + \sin^2\phi_1 d\phi_3^2. \tag{2.2}$$

In this parametrization, S^5 is given in terms of the five coordinates $(\psi, \theta, \phi_1, \phi_2, \phi_3)$. There are three obvious $U(1)$ isometries and they have to do with translations of the coordinates ψ, ϕ_2 and ϕ_3. On the gauge theory side each of them is in one-to-one correspondence with a $U(1)_R$ that rotates one of the three complex Higgs fields of the $\mathcal{N}=4$ theory. We denote them as Φ^1, Φ^2 and Φ^3. We make the correspondence

$$\Phi^1 \leftrightarrow J_{\Phi^1} = -i\partial_\psi, \tag{2.3}$$

$$\Phi^2 \leftrightarrow J_{\Phi^2} = -i\partial_{\phi_2}, \tag{2.4}$$

$$\Phi^3 \leftrightarrow J_{\Phi^3} = -i\partial_{\phi_3}. \tag{2.5}$$

In general, we would like to consider an arbitrary linear superposition of the three $U(1)$ isometries under which the complex fields Φ^1, Φ^2 and Φ^3 have charges Q_1, Q_2 and Q_3 respectively. The Penrose limit will be taken along a null geodesic associated to this isometry. For that purpose we introduce an angular coordinate ω' given by

$$-i\partial_{\omega'} \equiv -i(Q_1\partial_\psi + Q_2\partial_{\phi_2} + Q_3\partial_{\phi_3}) \tag{2.6}$$

and we suitably rescale it to get a new coordinate ω with periodicity 2π. Independently of the charges Q_1, Q_2 and Q_3, we can always write $\omega = \frac{\psi+\phi_2+\phi_3}{3}$ and the charge of every complex Higgs field, as measured by the current $-i\partial_\omega$, is one.

The geodesic of interest is given by

$$t = \omega,\ \rho = 0,\ \theta = \theta_0,\ \phi_1 = \frac{\pi}{4},\ \psi = \phi_2 = \phi_3 = \omega, \tag{2.7}$$

with $\theta_0 = \arccos(1/\sqrt{3})$. Indeed, a simple substitution of these values in (2.1) gives the null geodesic condition

$$ds^2 = R^2\Big(- dt^2 + d\omega^2\Big) = 0. \tag{2.8}$$

In order to focus on the geometry of the neighborhood of this geodesic we introduce new coordinates

$$x^+ = \frac{1}{2}(t+\omega), \tag{2.9}$$

$$x^- = \frac{R^2}{2}(t-\omega) \tag{2.10}$$

and perform the rescaling

$$\rho = \frac{r}{R},\ \theta = \theta_0 + \frac{y_1}{R},\ \phi_1 = \frac{\pi}{4} + \sqrt{\frac{3}{2}}\frac{y_2}{R},\ \psi = \omega - \sqrt{2}\frac{x_1}{R}, \tag{2.11}$$

$$\phi_2 = \omega + \frac{1}{\sqrt{2}}\frac{x_1 - \sqrt{3}x_2}{R},\ \phi_3 = \omega + \frac{1}{\sqrt{2}}\frac{x_1 + \sqrt{3}x_2}{R}, \tag{2.12}$$

taking the $R \to \infty$ limit. The numerical factors have been inserted for later convenience.

Expanding each expression in (2.1) up to second order in $1/R^2$ gives the pp-wave metric

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + \sum_{i=1}^{4} dr^i dr^i + \sum_{a=1,2}(dy_a^2 + dx_a^2 + 4y_a dx_a dx^+). \tag{2.13}$$

The full solution is also supported by the constant 5-form flux of eq.(1.5). Following [32] we will hereafter refer to this background as the magnetic pp-wave limit of $AdS_5 \times S^5$. It is a maximally supersymmetric background with 32 supersymmetries and its gauge theory dual is a suitable truncation of the $\mathcal{N} = 4$ SYM. This truncation is independent of the choice of the $U(1)_R$ and therefore it is not different from the one that appears in [16]. It is worth noticing that the same pp-wave background also appears in [31,32,33], where the Penrose limit was taken on $AdS^5 \times T^{1,1}$. The gauge theory dual in that case is an $\mathcal{N} = 1$ $SU(N) \times SU(N)$ SYM with a pair of bifundamental chiral multiplets A_i and B_i transforming in the $(N, \bar{N})$ and $(\bar{N}, N)$ representation of the gauge group. The fact that it can also be obtained from $AdS_5 \times S^5$ in the fashion that we discuss here was also mentioned in [32].

The correspondence between the light-cone momenta p^- and p^+ on the string theory side and the scaling dimensions and R-charges on the gauge theory side works in the following way

$$2p^- = -p_+ = i\partial_{x^+} = i(\partial_t + \partial_\omega) = \Delta - J \tag{2.14}$$

and

$$2p^+ = -p_- = i\partial_{x^-} = \frac{1}{R^2} i(\partial_t - \partial_\omega) = \frac{1}{R^2}(\Delta + J). \tag{2.15}$$

R is the radius of AdS_5 and we have set

$$J = -i\partial_\omega = \frac{1}{Q_1} R_1 + \frac{1}{Q_2} R_2 + \frac{1}{Q_3} R_3. \tag{2.16}$$

For each $i = 1, 2, 3$, R_i is a $U(1)$ generator under which only Φ^i is charged and the charge is Q_i.

In the limit under consideration $R \to \infty$. Since we only keep the states with finite p^+ it is necessary to take the familiar scaling $\Delta, J \sim R^2 \sim \sqrt{N}$. As a result, on the gauge theory side we must take the $N \to \infty$ limit keeping the Yang-Mills coupling fixed and small and focus on operators with large R-charge $J \sim \sqrt{N}$ and small and fixed $\Delta - J$. Such operators were introduced in [16] and we re-discuss them in the magnetic pp-wave context in section 2.3.

2.2. String propagation on magnetic pp-waves

The gauge-fixed light-cone bosonic string action for the background (2.13) is [32]

$$S=\frac{1}{2\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma\left(\frac{1}{2}\partial_a\vec{r}\partial^a\vec{r}-\frac{1}{2}r^2+\frac{1}{2}\partial_a\vec{x}\partial^a\vec{x}+\frac{1}{2}\partial_a\vec{y}\partial^a\vec{y}-2\vec{x}\partial_\tau\vec{y}\right). \quad (2.17)$$

There are several terms contributing to this action. There are four massive oscillators labeled by the 4-dimensional vector $\vec{r}$ and two identical decoupled Landau actions involving the coordinates (x_1, y_1) and (x_2, y_2). Each of them is precisely the action of a 2-dimensional charged particle moving in a constant magnetic field. It is convenient to rewrite the $x-y$ part of the action by performing the rotation

$$x_a=-\frac{1}{\sqrt{2}}(\hat{x}_a+\hat{y}_a),\quad y_a=\frac{1}{\sqrt{2}}(\hat{x}_a-\hat{y}_a). \quad (2.18)$$

Up to a total derivative term that can be dropped the action takes the form

$$S_{xy}=\frac{1}{2\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma\left(\frac{1}{2}\partial_a\vec{\hat{x}}\cdot\partial^a\vec{\hat{x}}+\frac{1}{2}\partial_a\vec{\hat{y}}\cdot\partial^a\vec{\hat{y}}-\vec{\hat{x}}\cdot\partial_\tau\vec{\hat{y}}+\vec{\hat{y}}\cdot\partial_\tau\vec{\hat{x}}\right) \quad (2.19)$$

and from now on we drop the $\hat{}$ notation. This action and the associated spectrum have also appeared in the context of the Penrose limit of $AdS_5\times T^{1,1}$ in [31]. For completeness, in the rest of this subsection we review the spectra that were obtained there.

The spectrum of the r^i part of the light-cone Hamiltonian reads

$$\mathcal{H}_r=\sum_{n=-\infty}^{\infty}N_n^{(r)}\sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}. \quad (2.20)$$

There are four kinds of oscillators contributing to the level $N_n^{(r)}$ and we denote them as a_n^i, for $i=1,2,3,4$. We use the notation of [16], so $n>0$ label the left movers and $n<0$ label the right movers.

For the $x-y$ part of the action the light-cone Hamiltonian breaks up into four parts

$$\mathcal{H}_{xy}=\sum_{n=-\infty}^{\infty}\sum_{a=1,2}\left[N_n^{(b^a)}\left(\sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}+1\right)+N_n^{(\bar{b}^a)}\left(\sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}-1\right)\right]. \quad (2.21)$$

Four types of oscillators contribute to each of the above terms. The oscillators $(b_n^1, \bar{b}_n^1)$ originate from the (x_1, y_1) part of the Lagrangian and contribute to the levels $N_n^{b^1}$ and $N_n^{\bar{b}^1}$ respectively and the oscillators $(b_n^2, \bar{b}_n^2)$ contribute to the levels $N_n^{b^2}$ and $N_n^{\bar{b}^2}$.

These spectra can be derived by straightforward calculation, or they can be deduced from the following slightly different point of view [31]. After the change of variables (2.18), we introduce the complex coordinates $z_a = x_a + iy_a$ and we bring the metric (2.13) into the form

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + \sum_{i=1}^{4} dr^i dr^i + \sum_{a=1,2} (dz_a d\bar{z}_a + i(\bar{z}_a dz_a - z_a d\bar{z}_a)dx^+). \quad (2.22)$$

This background can be transformed into the maximally supersymmetric pp-wave solution of [16] if we perform the x^+-coordinate dependent $U(1) \times U(1)$ rotation

$$z_a = e^{ix^+} w_a, \quad \bar{z}_a = e^{-ix^+} \bar{w}_a. \quad (2.23)$$

In view of (2.14) this translates to

$$\begin{aligned} \Delta - J &= i\partial_{x^+}|_{z_a} \\ &= i\partial_{x^+}|_{w_a} + \sum_a (w_a \partial_{w_a} - \bar{w}_a \partial_{\bar{w}_a}) = (\Delta - J)_{S^5} + J_1 + J_2, \end{aligned} \quad (2.24)$$

where J_1 and J_2 are $U(1)$ rotation charges in the $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ transverse planes respectively.

The spectra of eqs.(2.20),(2.21) can be reproduced from (2.24) by noticing that the bosonic oscillators have the following J_1, J_2 charges

$$\begin{array}{ll} a_n^i & J_1 = J_2 = 0, \quad i = 1,2,3,4 \\ b_n^1 & J_1 = 1, J_2 = 0, \\ \bar{b}_n^1 & J_1 = -1, J_2 = 0, \\ b_n^2 & J_1 = 0, J_2 = 1, \\ \bar{b}_n^2 & J_1 = 0, J_2 = -1. \end{array} \quad (2.25)$$

The fermionic oscillator contributions to the light-cone Hamiltonian p^- can be similarly deduced from (2.24) by looking at the $U(1) \times U(1)$ charges carried by the $SO(8)$ spinor $\mathbf{8}_s$ under the $SU(2) \times SU(2) \times U(1) \times U(1)$ into which $SO(8)$ has been broken [31]

$$\mathbf{8}_s \to (\mathbf{2},\mathbf{1})_{(1/2,1/2)} \oplus (\mathbf{2},\mathbf{1})_{(-1/2,-1/2)} \oplus (\mathbf{1},\mathbf{2})_{(1/2,-1/2)} \oplus (\mathbf{1},\mathbf{2})_{(-1/2,1/2)}. \quad (2.26)$$

We get the spectra

$$
\begin{aligned}
S_n^{\alpha++} \quad 2p^- &= \sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}+1, \\
S_n^{\alpha--} \quad 2p^- &= \sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}-1, \\
S_n^{\dot\alpha+-} \quad 2p^- &= \sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2}, \\
S_n^{\dot\alpha-+} \quad 2p^- &= \sqrt{1+\left(\frac{n}{\alpha' p^+}\right)^2},
\end{aligned}
\tag{2.27}
$$

which, as expected, turn out to be identical to the bosonic ones.

Notice that the action of the bosonic zero mode oscillators $\bar{b}_0^1$ and $\bar{b}_0^2$, as well as the action of their fermionic superpartners $S_0^{\alpha--}$ has no effect on the light-cone energy. As a result, the spectrum exhibits an infinite degeneracy. The degenerate states are obtained by the action of an arbitrary number of the above zero mode oscillators on the vacuum. This degeneracy is familiar, since the worldsheet action contains two decoupled Landau parts, which describe a charged particle moving in the presence of a constant magnetic field in $\mathbf{R}^2 \times \mathbf{R}^2$. This system is known to have an infinite degeneracy of states labeled by the angular momentum of the charged particle.

In the next section we discuss how these bulk characteristics manifest themselves on the dual gauge theory.

2.3. The gauge/string correspondence

Now we would like to discuss the correspondence between the string oscillator states of the previous section and appropriate operators in the dual $\mathcal{N}=4$ SYM theory. Following [16] we are interested in the large N limit with g_{YM}^2 kept fixed and small. We work in the planar limit and examine single trace operators, which we categorize by their $\Delta - J$ value. As in the usual BMN limit there exists a very interesting finite J version of these operators [34,30], which we do not discuss in this paper.

We begin with single trace operators of $\Delta - J = 0$. There is an infinite number. Any traceless operator of the form $\mathrm{Tr}[\Phi^1 ... \Phi^2 ... \Phi^3 ...]$ containing J symmetrized insertions of the Φ^1, Φ^2 or Φ^3 fields has $\Delta - J = 0$. Each of them is an $\mathcal{N}=4$ chiral primary and its scaling dimension is protected by supersymmetry.

In order to construct the correspondence of SYM operators with string oscillator states, it is perhaps natural to single out a specific linear superposition of the Higgs fields associated to the $U(1)_R$ generator J that appears in (2.16). We choose the diagonal superposition

$$\Omega = \frac{1}{\sqrt{3}}(\Phi^1 + \Phi^2 + \Phi^3). \tag{2.28}$$

In the language of [16] we propose the correspondence

$$\frac{1}{\sqrt{J}N^{J/2}}\mathrm{Tr}[\Omega^J] \leftrightarrow |0, p^+; \sigma_\Omega\rangle_{l.c.}, \tag{2.29}$$

where σ_Ω is a formal parameter that denotes a particular state of the infinitely degenerate light-cone vacuum space. The factor of the l.h.s. is such that the normalization of the two point function is one.

To obtain the rest of the $\Delta - J = 0$ operators we act on the above vacuum with an arbitrary number of the zero mode oscillators $\bar{b}_0^1, \bar{b}_0^2$. Since they have no effect on the light-cone energy, these oscillators should be associated again to linear combinations of the Higgs fields Φ^1, Φ^2 and Φ^3. We choose the two linear combinations that are orthogonal to Ω and propose the correspondence

$$\bar{b}_0^1 \leftrightarrow \Psi^1 = \frac{1}{\sqrt{2}}(\Phi^2 + \Phi^3 - 2\Phi^1) \tag{2.30}$$

and

$$\bar{b}_0^2 \leftrightarrow \Psi^2 = \frac{1}{\sqrt{6}}(\Phi^3 - \Phi^2). \tag{2.31}$$

It is clear that the above correspondence between operator insertions and string oscillators is by no means unique. Any $SU(3)$ rotated basis of Higgs fields could equally well be assigned to the same string oscillators. This lack of uniqueness is also manifest on the arbitrary choice of the state $|0, p^+; \sigma_\Omega\rangle_{l.c.}$ on the r.h.s. of (2.29).

With the above correspondence the action of the zero mode oscillators $\bar{b}_0^{a\dagger}$ $(a = 1, 2)$ on the light-cone vacuum (2.29) can be translated in the SYM language as follows. For each $\bar{b}_0^{a\dagger}$ we are instructed to make an insertion of Ψ^a and then sum over all possible orderings. This is the same as acting on $\mathrm{Tr}[\Omega^J]$ with the operator $\sum_{l=1}^{J}(\Omega\Psi^a\frac{\partial}{\partial\Omega})_l$, where we use the notation $(...)_l$ to denote that the operator in parenthesis acts on the lth insertion of the trace. For example,

$$\frac{1}{\sqrt{J}}\sum_l \frac{1}{\sqrt{J}N^{J/2+1/2}}\mathrm{Tr}[\Omega^l\Psi^a\Omega^{J-l-1}] \leftrightarrow \bar{b}_0^{a\dagger}|0, p^+; \sigma_\Omega\rangle_{l.c.}. \tag{2.32}$$

Repeated action of these zero modes creates the anticipated Landau degeneracy of the vacuum, which becomes infinite in the $J \to \infty$ limit.

For the operators with $\Delta - J = 1$ we can say the following. There are *twelve* bosonic operators of this type, $D_i\Omega$, $D_i\Psi^1$ and $D_i\Psi^2$ and they are expected to match the four zero mode oscillators a_0^i for $i = 1, 2, 3, 4$. This correspondence works by associating

$$a^{i\dagger} \leftrightarrow \sum_l (\Omega D_i)_l \quad i = 1, 2, 3, 4. \tag{2.33}$$

D_i denotes the gauge covariant derivative with respect to the spacetime coordinates of $\mathbf{R}^4$ where the dual $\mathcal{N} = 4$ gauge theory lives. More precisely, whenever we act on the vacuum $|0, p^+; \sigma_\Omega\rangle_{l.c.}$ of eq.(2.29) with the oscillator $a^{i\dagger}$, we are instructed to add an insertion of $D_i\Omega$ on the gauge theory operator $\mathrm{Tr}[\Omega^J]$ and then sum over all possible orderings, e.g.

$$\frac{1}{\sqrt{J}} \sum_{l=0}^{J} \frac{1}{\sqrt{J} N^{J/2+1/2}} \mathrm{Tr}[\Omega^l D_i \Omega \Omega^{J-l}] \leftrightarrow a_0^{i\,\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}. \tag{2.34}$$

Acting on a different vacuum state of the same light-cone energy, e.g. acting on $\bar{b}_0^{a\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}$, also amounts to a similar insertion of $D_i\Omega$ or $D_i\Psi^a$. We insert $D_i\Omega$ if a position is initially occupied by Ω and $D_i\Psi^a$ if the position is initially occupied by Ψ^a. This rule is a consequence of the fact that the state $a_0^{i\,\dagger} \bar{b}_0^{a\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}$ can also be written as $\bar{b}_0^{a\dagger} a_0^{i\,\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}$.

Finally, we have to consider insertions of the $\Delta - J = 2$ operator $\bar{\Psi}^a$. From the string spectrum (2.21) it is apparent that such insertions correspond to the action of the zero mode oscillators $b_0^{a\,\dagger}$, which increase the light-cone Hamiltonian by 2. It is therefore natural to make the identification

$$\frac{1}{\sqrt{J}} \sum_l \frac{1}{\sqrt{J} N^{J/2+1/2}} \mathrm{Tr}[\Omega^l (\bar{\Psi}^a) \Omega^{J-l}] \leftrightarrow b_0^{a\,\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}. \tag{2.35}$$

The above correspondence also extends nicely to the fermionic zero mode oscillators (2.27). The relevant SYM operators follow easily from the bosonic ones by supersymmetry. We have

$$\begin{array}{ccc} \textbf{gauge theory fermionic operators} & \leftrightarrow & \textbf{fermionic string oscillators} \\ \bar{\lambda}^{\dot{\alpha}+-} & & S^{\dot{\alpha}+-}, \\ \bar{\lambda}^{\dot{\alpha}-+} & & S^{\dot{\alpha}-+}, \\ \bar{\psi}^1 & & S^{1++}, \\ \bar{\psi}^2 & & S^{2++}, \\ \psi^1 & & S^{1--}, \\ \psi^2 & & S^{2--}. \end{array} \tag{2.36}$$

$\bar{\lambda}$ denotes the right-handed gauginos. There are 8 such components. Each of them has a definite charge ($\pm 1/2$) under the two "Landau" $U(1)$'s into which $SO(4) \subset SO(6)_R$ has been broken. The $+/-$ superscripts denote the components of the gauginos with charges $\pm 1/2$ respectively. ψ^a for $a = 1, 2$ are the fermionic superpartners of the bosons Ψ^a.

For the higher excited modes of the string the correspondence works exactly as in [16]. The action of any excited oscillator is expressed in the SYM language by the insertion of the corresponding field multiplied by a position dependent phase, e.g.

$$\frac{1}{\sqrt{J}} \sum_l \frac{1}{N^{J/2+1}} \mathrm{Tr}[\Psi^a \Omega^l \Psi^b \Omega^{J-l}] e^{\frac{2\pi i n l}{J}} \leftrightarrow \bar{b}_n^{b\dagger} \bar{b}_{-n}^{a\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.}. \tag{2.37}$$

The details of this construction are precisely the same as in [16] and we will not discuss them further.

In conclusion, we rephrased the BMN correspondence at the $\mathcal{N} = 4$ SYM fixed line for a diagonal $U(1)_R$ choice. We did not go into much detail, because the essence of the correspondence is expected to be independent of this choice and in particular, it should be easy to translate all the checks and extensions of the correspondence at finite J in the language of this section. Furthermore, it is natural to expect that this same BMN correspondence also persists when we deform away from the $\mathcal{N} = 4$ fixed line. The goal of the next section is to determine the effect of the deformation on the BMN operators.

3. $\mathcal{N} = 1$ superconformal theories and BMN operators

3.1. Exactly marginal deformations of the $\mathcal{N} = 4$ SYM theory

After this long parenthesis on magnetic pp-waves, we are now ready to proceed with the analysis of the Leigh-Strassler deformations of the $\mathcal{N} = 4$ SYM theory. The four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory can be expressed in the language of $\mathcal{N} = 1$ supersymmetry in terms of a vector multiplet V and three chiral multiplets[1] Φ^i, $i = 1, 2, 3$. In addition to the usual kinetic terms of the $\mathcal{N} = 1$ theory one is also instructed to add a superpotential of the form

$$W = g' \mathrm{Tr}([\Phi^1, \Phi^2]\Phi^3). \tag{3.1}$$

In this $\mathcal{N} = 1$ language only an $SU(3) \times U(1)$ subgroup of the full $SU(4)_R$ R-symmetry group is manifest. $SU(3)$ is the group that rotates the chiral superfields Φ^i. At the $\mathcal{N} = 4$

[1] In this section Φ^i, Ω and Ψ^a denote full $\mathcal{N} = 1$ superfields and they should not be confused with the bosonic bottom components of the previous section.

point the superpotential coupling g' is directly related to the Yang-Mills coupling and in our conventions $g' = \sqrt{2} g_{YM}$. To set our notation straight we write the full $\mathcal{N} = 4$ action as

$$\mathcal{S} = \mathrm{Tr}\Big(\int d^4\theta e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i + \frac{1}{2g^2} \Big[\int d^4x d^2\theta W^\alpha W_\alpha + \int d^4x d^2\bar{\theta} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \Big] + \\ + \frac{g'}{3!} \int d^4x d^2\theta \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] - \frac{g'}{3!} \int d^4x d^2\bar{\theta} \epsilon^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \Big) \tag{3.2}$$

and by definition we always set $g = \sqrt{2} g_{YM}$. Notice the explicit distinction between the superpotential coupling g' and the vector superfield coupling g. At the $\mathcal{N} = 4$ fixed line we have $g = g'$ but this relation is modified as we deform away and in general we need to differentiate between the two couplings.

Since the $\mathcal{N} = 4$ theory is conformal for any value of the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$, the deformation that changes this value is obviously exactly marginal. It is also known, however, that for $N \geq 3$ the $\mathcal{N} = 4$ theory has additional exactly marginal perturbations [15]. Classically, one possibility is given by the superpotential

$$W = h_{ijk} \mathrm{Tr}(\Phi^i \Phi^j \Phi^k), \tag{3.3}$$

with ten symmetric coefficients h_{ijk}. Another one is the superpotential (3.1) with any (complex) coefficient g'. For the first class, it is known [12,13,15] that only a two-complex parameter subset of them is exactly marginal on the quantum level. The resulting superpotential can be written as

$$W_{def} = h_1 \mathrm{Tr}(\Phi^1\Phi^2\Phi^3 + \Phi^1\Phi^3\Phi^2) + h_2 \mathrm{Tr}((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3), \tag{3.4}$$

in terms of two complex coefficients h_1, h_2. These particular deformations preserve a $Z_3 \times Z_3$ symmetry given by the transformations $\Phi^1 \to \Phi^2$, $\Phi^2 \to \Phi^3$, $\Phi^3 \to \Phi^1$ and $\Phi^1 \to \Phi^1$, $\Phi^2 \to \omega\Phi^2$, $\Phi^3 \to \omega^2\Phi^3$. ω is a cubic root of unity. The second Z_3 prevents any mixing between the chiral operators Φ^i and the first can be used to show that they all have the same anomalous dimension $\gamma(\tau, g', h_1, h_2)$. The beta functions are restricted by non-renormalization theorems to be proportional to this anomalous dimension and the constraint

$$\gamma(g, g', h_1, h_2) = 0 \tag{3.5}$$

gives a 3-complex dimensional surface of fixed points. For simplicity, we set the theta angle to zero.

The analytic form of this surface is only known up to first order in perturbation theory [12,35,36]. Notice that for generic points in this moduli space the coefficient g' is not necessarily equal to the $\mathcal{N}=4$ value $g=\sqrt{2}g_{YM}$. It turns out that the large R-charge limit, on which we base our analysis, probes a neighborhood of this moduli space around the strong 't-Hooft coupling point. Thus, for later considerations it is convenient to write g' as $g'=g+h_0$, with h_0 complex. At the end of the day, our results on the anomalous dimensions of the BMN operators will be expressed in terms of the three independent couplings g, h_1 and h_2.

The conclusion of this short introduction is that for fixed g there are basically two exactly marginal deformations away from the $\mathcal{N}=4$ fixed line and they correspond to the superpotential (3.4). On the supergravity side this deformation can be identified at first order with part of the KK scalar mode in the **45** of $SO(6)$ [6,37]. This scalar corresponds to the second two-form harmonic $Y^I_{[\alpha,\beta]}$ in the expansion of the complex antisymmetric two-form $A_{\alpha,\beta}$ with components along the five-sphere. The effect of the deformation in supergravity has been analyzed perturbatively in the deformation parameters in [6,12] and is expected to be a warped fibration of AdS_5 over a deformed $\tilde{S}^5$ in the presence of 3-form and 5-form fluxes. An interesting class of supergravity solutions of this type was also obtained in [11]. These solutions, however, appear to be singular and their exact relation to the deformation superpotential (3.4) is not clear.

3.2. BPS and near-BPS operators

In section 2 and in the context of a "magnetic" Penrose limit of $AdS_5\times S^5$ we considered a class of large R-charge operators of the $\mathcal{N}=4$ SYM theory, which were obtained from the operator

$$\Pi_J \equiv \frac{1}{\sqrt{J}N^{J/2}}\mathrm{Tr}[\Omega^J] \tag{3.6}$$

by insertions of the fields $D_i\Omega$, Ψ^a and $\bar{\Psi}^a$ ($i=1,2,3,4$ and $a=1,2$) with or without position dependent phases. Without such phases the resulting symmetrized operators are 1/2-BPS. They are protected operators of the $\mathcal{N}=4$ theory because they belong to short multiplets of the $SU(2,2|4)$ superconformal group[2] .

[2] More specifically, they are protected because they belong to short multiplets that cannot combine to form long multiplets after the $\mathcal{N}=4$ interaction is turned on. See e.g. [38] for a recent discussion on this point.

Alternatively, we can ask in what sense they are protected from an $\mathcal{N}=1$ point of view. Generically an $\mathcal{N}=4$ short multiplet can break into $\mathcal{N}=1$ short and long multiplets and it is not immediately obvious how the $\mathcal{N}=4$ protection manifests itself in the $\mathcal{N}=1$ formalism. This question is even more important and instructive in anticipation of the Leigh-Strassler deformation that breaks the $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$. We need to know what remains protected even after the deformation. The $\mathcal{N}=1$ of interest is the one that is preserved by the Leigh-Strassler deformations, i.e. one under which all three Higgs fields have equal R-charge 2/3.

Let us first see what happens along the $\mathcal{N}=4$ fixed line from an $\mathcal{N}=1$ point of view. Π_J is protected, because it is an $\mathcal{N}=1$ chiral primary operator and obeys the BPS condition $\Delta=J$. The same is also true for the operators that arise when we include symmetrized insertions of the fields Ψ^a. Insertions of the fields $D_i\Omega$ lead to descendants of Π_J and they are also protected. The remaining operators are those with $\bar{\Psi}^a$ insertions. Every such insertion has $\Delta-J=2$ at weak coupling and clearly does not produce an $\mathcal{N}=1$ chiral field. Nevertheless, the resulting operator is still $\mathcal{N}=1$ protected, because it belongs to another type of short multiplet of $SU(2,2|1)$ and in $\mathcal{N}=1$ notation it is known as a semi-conserved superfield (see, for example, [39]). Semi-conserved superfields L obey the condition[3]

$$\bar{D}^2 L = 0. \tag{3.7}$$

Using the $\mathcal{N}=4$ SYM equations of motion one can easily verify that the corresponding superfields with $\bar{\Psi}^a$ insertions indeed satisfy this condition.

On the other hand, operators with the above insertions and position-dependent phases are not protected, because the insertions are not symmetrized. For example, operators of the type

$$\sum_l e^{\frac{2\pi i n}{J}} \mathrm{Tr}[\Psi^a\Omega^l\Psi^b\Omega^{J-l}] \tag{3.8}$$

have $\Delta-J=0$ at weak coupling and they may seem to be chiral and hence protected. This, however, is not correct, because one can use the $\mathcal{N}=4$ SYM equations of motion to symmetrize this operator. In the process extra terms appear and they turn out to be descendants of non-protected operators. A similar reasoning can also be applied to other non-symmetrized operators.

[3] D_α and $\bar{D}_{\dot{\alpha}}$ are the usual superspace covariant derivatives. In what follows, we work in $\mathcal{N}=1$ superspace and adopt the notations of [40].

Once we deform the $\mathcal{N}=4$ SYM action by the superpotential (3.4) at a generic point of the moduli manifold (3.5) many of the above statements about the 0-level BPS operators change. As we verify explicitly in the next section, the deformation modifies the $\mathcal{N}=4$ equations of motion and the previously protected operators acquire nonzero anomalous dimensions. For example, it is easy to check that (3.7) breaks down away from the $\mathcal{N}=4$ point and operators with $\bar{\Psi}^a$ insertions no longer remain semi-conserved in the deformed theories. Similarly, the previously symmetrized chiral operators with Ψ^a insertions acquire anomalous dimensions and they are not protected against the $\mathcal{N}=4$-breaking deformations. These anomalous dimensions are computed in the next section using the technology of [17] and they are verified independently to leading order in perturbation theory in section 3.3.

Only one operator remains protected and continues to have $\Delta - J = 0$. This is Π_J. The vanishing of its anomalous dimension is synonymous to the condition (3.5) that guarantees the presence of superconformal invariance in the deformed theory. As a result, we see that the effect of the deformation is to lift the infinite Landau degeneracy of the $\mathcal{N}=4$ point and retain a single vacuum state represented on the gauge theory side by the operator Π_J. Such a vacuum state with vanishing light-cone energy should also be expected from the supersymmetry of the dual background.

Another aspect of this picture is the following. We have concentrated our attention on the BMN operators that can be obtained from Π_J by appropriate insertions of other fields and worked mainly in a "dilute gas" approximation. In doing so, we break the Z_3 symmetry that permutes the three adjoint chiral superfields and the "vacuum" operators $\mathrm{Tr}[(\Psi^1)^J]$ and $\mathrm{Tr}[(\Psi^2)^J]$ remain at "infinite distance" from the operator Π_J, i.e. they result from infinite insertions. This seems to be inconsequential for the BMN correspondence at the $\mathcal{N}=4$ point, because of the infinite Landau degeneracy, but it is perhaps a little puzzling for the BMN correspondence after the deformation. These operators have similar properties as Π_J and they continue to have $\Delta - J = 0$ throughout the moduli space. In order to obtain them from Π_J we have to start adding insertions that increase the total $\Delta - J$ [4] and it is not completely obvious how we can recover an operator with $\Delta - J = 0$. The key point has to be that after several insertions the "dilute gas" approximation starts breaking down and one has to be more careful on the derivation of the scaling dimensions. This process is also obscured by the fact that we have to add an infinite number of insertions and this is not something completely well-defined.

[4] For the type of $\Delta - J$ values that we find after the deformation, see for example Table 1.

In order to calculate the anomalous dimensions of the above operators, we would like to determine the appropriate two-point functions. The authors of [17] performed a similar calculation at the $\mathcal{N}=4$ point by working in superspace formalism and using the constraint imposed by the equations of motion of the theory.[5] Following their example, we consider the operators[6]

$$\mathcal{U}^a_J = \sum_l e^{il\varphi}\Omega^l \bar{\Psi}^a \Omega^{J-l} \tag{3.9}$$

and

$$\mathcal{O}^a_J = \sum_l e^{il\varphi}\Omega^l \Psi^a \Omega^{J-l}, \tag{3.10}$$

for $a = 1, 2$ and $\varphi = \frac{2\pi n}{J}$. The actual operators that appear in the BMN construction are traced gauge invariant operators of the type

$$\sum_l e^{il\varphi}\mathrm{Tr}[\Psi^a \Omega^l \Psi^b \Omega^{J-l}]. \tag{3.11}$$

They contain the above $\mathcal{U}^a_J$ and $\mathcal{O}^a_J$ as "building blocks" and under the "dilute gas" approximation the latter are the dominant pieces in the calculation of the anomalous dimensions.

In the presence of the deformations the gauge theory equations of motion become

$$\begin{aligned} \frac{1}{4}\bar{D}^2\bar{\Psi}^1 &= g'[\Psi^2, \Omega] + h_1\{\Psi^2, \Omega\} + 3h_2(\Psi^1)^2, \\ \frac{1}{4}\bar{D}^2\bar{\Psi}^2 &= -g'[\Psi^1, \Omega] + h_1\{\Psi^1, \Omega\} + 3h_2(\Psi^2)^2. \end{aligned} \tag{3.12}$$

Notice that the gauge theory action has been expressed in terms of the rotated basis of superfields (Ω, Ψ^1, Ψ^2). This is not necessary, but we do it here in order to comply with the conventions adopted in section 2.3. In the large J limit the above equations imply

$$\begin{aligned} \frac{1}{4}\bar{D}^2\mathcal{U}^1_J &= (g'(1-e^{-i\varphi}) + h_1(1+e^{-i\varphi}))\mathcal{O}^2_{J+1} + 3h_2\mathcal{O}^{11}_J, \\ \frac{1}{4}\bar{D}^2\mathcal{U}^2_J &= (-g'(1-e^{-i\varphi}) + h_1(1+e^{-i\varphi}))\mathcal{O}^1_{J+1} + 3h_2\mathcal{O}^{22}_J, \end{aligned} \tag{3.13}$$

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[5] A similar calculation for $\mathcal{N}=2$ superconformal gauge theories based on ADE quiver diagrams was performed in [41].

[6] Gauge invariance demands that the operator $\mathcal{U}^a_J$ should be written as $\sum_l e^{il\varphi}\Omega^l e^{-gV}\bar{\Psi}^a e^{gV}\Omega^{J-l}$. For our purposes, however, it is enough to work with the assumption that $V=0$.